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Whitney preserving maps on finite graphs

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ABSTRACT

For a Whitney preserving map $f : X \to G$ we show the following: (a) If X is arcwise connected and G is a graph which is not a simple closed curve, then f is a homeomorphism; (b) If X is locally connected and G is a simple closed curve, then X is homeomorphic to either the unit interval $[0,1]$, or the unit circle S^1. As a consequence of these results, we characterize all Whitney preserving maps between finite graphs. We also show that every hereditarily weakly confluent Whitney preserving map between locally connected continua is a homeomorphism.

1. Introduction

In [2] it was proved that any Whitney preserving map from an arcwise connected continuum onto the unit interval must be a homeomorphism. In this paper, Section 3, we extend this result by showing that any Whitney preserving map from an arcwise connected continuum onto a finite graph, which is not a simple closed curve, is a homeomorphism, see 3.11. Asking the image not to be a simple closed curve is necessary since there exists a Whitney preserving map from the unit interval to the unit circle S^1, see 2.3. As a corollary of 3.11 we have that the only Whitney preserving maps between a finite graph and another finite graph which is not a simple closed curve are homeomorphisms. We finish Section 3 showing that there exist a continuum X, containing a dense arcwise connected subset, a graph G, admitting an Eulerian path and a Whitney preserving map $f : X \to G$ which is not a homeomorphism; thus showing that [2, Theorem 16], see 2.8, does not hold if we replace the unit interval with a graph that admits an Eulerian path; in particular, a simple closed curve.

In view of 2.3, it is natural to ask what conditions can be imposed to X such that a similar result to 3.11 can be achieved when the image of f is a simple closed curve. In Section 4, we show that if X is locally connected and there is a Whitney preserving map from X onto S^1, then X is homeomorphic to either the unit interval or to S^1. We combine the results of Sections 3 and 4 to characterize Whitney preserving maps between graphs. We also show, see 4.7, that a hereditarily

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weakly confluent Whitney preserving map from an arcwise semi-locally-connected continuum to a continuum Y must be a homeomorphism. As a consequence of this we have that every hereditarily weakly confluent Whitney preserving map between locally connected continua is a homeomorphism, see 4.8.

2. Definitions

In this section we introduce the definitions and the notation used throughout this paper. Also, for convenience to the reader, we state the theorems cited in our results. Most of these propositions can be found in [2,3,5,6].

Notation. *Let A and B be sets in a topological space such that $B \subseteq A$. Then $A - B$ denotes the set of all elements of A which are not elements of B, $\operatorname{int}_A(B)$ denotes the interior of B in A, $\operatorname{cl}_A(B)$ denotes the closure of B in A, $\operatorname{Fr}_A(B)$ denotes the boundary of B in A and $\operatorname{diam}(B)$ denotes the diameter of B.*

A *continuum* is a nondegenerate compact connected metric space. All spaces considered in this paper are continua. Given any continuum X, the symbol $C(X)$ denotes the *hyperspace of nonempty subcontinua* of X and the symbol $F_1(X)$ denotes the *hyperspace of singletons* of X, both with the topology generated by the Hausdorff metric, see [3] or [5].

A *finite graph* is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end points. For a given natural number $n \geqslant 3$, a *simple n-od* is the union of n arcs having only one end point in common, called the *vertex*. A simple 3-od is called a *simple triod*, see [6].

Throughout this paper the symbol I will denote the unit interval $[0,1]$ and the symbol S^1 will denote the unit circle, that is, $S^1 = \{z \in \mathbb{C}\colon |z| = 1\}$.

Given a function $f : X \to Y$ and a subset E of X, $f_{|E} : E \to Y$ denotes the restriction of f to E. The word *map* will be used as a synonym of continuous function. A map $f : X \to Y$ is called *weakly confluent* if for each subcontinuum C of Y there is a component A of $f^{-1}(C)$ such that $f(A) = C$. A map $f : X \to Y$ is called *hereditarily weakly confluent* if for every subcontinuum A of X, $f_{|A} : A \to f(A)$ is weakly confluent.

Every map $f : X \to Y$ induces a continuous function $\widehat{f} : C(X) \to C(Y)$ given by

$$\widehat{f}(A) = f(A) \quad \text{for all } A \in C(X).$$

It is known that f is weakly confluent if and only if $\widehat{f}$ is onto.

2.1. Definition. A continuous function $\mu : C(X) \to \mathbb{R}$ is called a *Whitney map* if it satisfies the following two conditions:

(i) $\mu(\{x\}) = 0$ for all $x \in X$ and
(ii) if $A, B \in C(X)$ are such that $A \subseteq B$ and $A \neq B$, then $\mu(A) < \mu(B)$.

Let $\mu : C(X) \to \mathbb{R}$ be a Whitney map and let $t \in [0, \mu(X)]$, the set $\mu^{-1}(t)$ is called a *Whitney level of* $C(X)$.

2.2. Definition. Let $f : X \to Y$ be a continuous function between continua. The map f is called *Whitney preserving* if there exist two Whitney maps $\mu : C(X) \to \mathbb{R}$ and $\nu : C(Y) \to \mathbb{R}$ such that for every $t \in [0, \mu(X)]$ there exists $s \in [0, \nu(Y)]$ such that

$$\widehat{f}\big(\mu^{-1}(t)\big) = \nu^{-1}(s).$$

In this case we say that f is μ, ν-Whitney preserving.

2.3. Example. Let $f : [0,\pi] \to S^1$ be the map given by $f(x) = e^{4xi}$. If $\mu : C([0,\pi]) \to \mathbb{R}$ is the diameter map and $\nu : C(S^1) \to \mathbb{R}$ is the map that assigns to each subcontinuum of S^1 its arc-length, then f is μ, ν-Whitney preserving. To see this, note that the map f wraps the interval $[0,\pi]$ around S^1 twice in such a way that every two arcs of $[0,\pi]$ with the same length (less than $\frac{\pi}{2}$) are mapped to two arcs of S^1 with the same length, and any arc with length greater than or equal to $\frac{\pi}{2}$ is mapped onto S^1.

A consequence of the previous definition is that every Whitney preserving map is weakly confluent.

2.4. Theorem. *(Proposition 9 of [2].) Let X and Y be continua. If $f : X \to Y$ is a Whitney preserving map, and for some subcontinuum Z of X, $f_{|Z} : Z \to f(Z)$ is weakly confluent, then $f_{|Z}$ is Whitney preserving.*

2.5. Remark. From 2.4 we have that if $f : X \to Y$ is a hereditarily weakly confluent Whitney preserving map, then $f_{|Z}$ is Whitney preserving for every subcontinuum Z of X.

2.6. Lemma. *Let $f : X \to Y$ be a Whitney preserving map. If Z is a subcontinuum of X such that $f(Z)$ is an arc, then $f_{|Z} : Z \to f(Z)$ is Whitney preserving.*

Proof. It is known that every map from any continuum onto an arc-like continuum is weakly confluent (see [6, 12.46]). Since $f(Z)$ is an arc, $f(Z)$ is an arc-like continuum. Then, from 2.4, we have that $f_{|Z}: Z \to f(Z)$ is Whitney preserving. $\square$

2.7. Definition. A subset A of a continuum X is said to be an *arc component* if it is a maximal arcwise connected subset of X. If, in addition, A is dense in X, then it is said to be a *dense arc component of* X.

2.8. Theorem. *(Theorem 16 of [2].) Let X be a continuum such that X contains a dense arc component. If $f: X \to I$ is Whitney preserving, then f is a homeomorphism.*

3. Maps onto graphs

In this section, G will denote a finite graph and all Whitney maps $\mu: C(X) \to \mathbb{R}$ will be such that $\mu(X) = 1$.

3.1. Lemma. *Let $f: X \to G$ be a Whitney preserving map. There exists $\delta > 0$ such that if A is an arc in X and* $\operatorname{diam}(A) < \delta$, *then $f_{|A}$ is one-to-one or $f(A)$ is degenerate.*

Proof. Let $\varepsilon > 0$ be such that for each edge J of G, $\varepsilon < \operatorname{diam}(J)$. Let $\delta > 0$ be given by ε and the uniform continuity of f. Let A be an arc in X with $\operatorname{diam}(A) < \delta$. Assume $f(A)$ is nondegenerate, we will show that $f_{|A}: A \to f(A)$ is one-to-one. If $f(A)$ is an arc, then $f_{|A}$ is weakly confluent and, by 2.4 and 2.8, $f_{|A}$ is Whitney preserving and a homeomorphism, hence it is one-to-one. Now, assume that $f(A)$ is nondegenerate and not an arc. Since $f(A)$ contains no edge of G, $f(A)$ is a finite graph that has at most one vertex and contains no simple closed curve. Therefore, $f(A)$ is a simple m-od, with $m \geqslant 3$. Hence, $f(A) = J_1 \cup \cdots \cup J_m$ and there is a vertex v of G such that for each $i, j \in \{1, 2, \ldots, m\}$, J_i is an arc having v as an end point and $J_i \cap J_j = \{v\}$ for $i \neq j$.

Let p, q be the end points of A. Since the set $f(\{p, q\})$ has at most two points, we can assume that $f(\{p, q\}) \cap (J_1 - \{v\}) = \emptyset$. Let $x \in A$ be such that $f(x) \in J_1 - \{v\}$. Assume that the order on A, induced by its topology, satisfies $p < x < q$. Let $u = \max\{s \in [p, x]\colon f(s) \in J_2 \cup \cdots \cup J_m\}$ and $w = \min\{s \in [x, q]\colon f(s) \in J_2 \cup \cdots \cup J_m\}$. Notice that $u < x < w$, $f(u) = v = f(w)$ and $f(s) \in J_1 - \{v\}$ for all $s \in (u, w)$. Therefore, $f([u, w])$ is a subarc of J_1 with $\operatorname{diam}([u, w]) < \delta$. Using an argument similar to the one on the previous paragraph we have that $f_{|[u,w]}: [u, w] \to f([u, w])$ is a homeomorphism, this is a contradiction to the fact that $f(u) = f(w)$. Hence $f(A)$ cannot be a simple m-od for any $m \geqslant 3$. $\square$

3.2. Lemma. *Let $f: X \to Y$ be a Whitney preserving map between continua and let α be an arc in X. If f is not light, then $f(\alpha)$ is degenerate.*

Proof. Suppose that f is not light. Let $f: X \to Y$ be a μ, ν-Whitney preserving map and A be a nondegenerate subcontinuum of X such that $f(A)$ is degenerate. Let $t = \mu(A) > 0$, then $\widehat{f}(\mu^{-1}(t)) = F_1(Y)$. Choose $\delta > 0$ such that if $B \in C(X)$ and $\operatorname{diam}(B) < \delta$, then $\mu(B) < t$. Now, α can be written as $\alpha = \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n$, where each α_i is an arc and $\operatorname{diam}(\alpha_i) < \delta$. For each $i \in \{1, \ldots, n\}$, since $\mu(\alpha_i) < t$, there is a subcontinuum A_i of X such that $\alpha_i \subseteq A_i$ and $\mu(A_i) = t$. Therefore $f(A_i)$ is degenerate; this implies that $f(\alpha)$ is degenerate. $\square$

The following corollary generalizes Lemma 15 of [2]. Notice that the corollary is very similar to Lemma 2.3 of [4]; however, we do not require the dense arcwise connected set to be maximal.

3.3. Corollary. *Let $f: X \to Y$ be a Whitney preserving map between continua. Assume Y is nondegenerate and that X contains a dense arcwise connected set. Then f is light.*

3.4. Definition. Let $f: X \to G$ be a continuous function and let K be a nondegenerate subcontinuum of X. We say that $f_{|K}$ is *almost one-to-one* if $(f_{|K})^{-1}(y)$ is finite for all $y \in G$ and $\{y \in G\colon (f_{|K})^{-1}(y)$ is nondegenerate$\}$ is finite. A continuous, almost one-to-one, onto map, $g: \alpha \to G$ from an arc α with end points u and v, is called an *Eulerian path* if $g(u) = g(v)$.

3.5. Lemma. *Let $f: X \to G$ be a continuous function and $K \in C(X)$ such that $f_{|K}$ is almost one-to-one. Then K is a finite graph.*

Proof. Let $p \in K$. First we will construct a compact neighborhood M_p of p in K as follows. Let $F = \{y \in G\colon (f_{|K})^{-1}(y)$ is nondegenerate$\}$. If $f(p) \notin F$, since F is closed in G, there is a compact neighborhood M_p of p in K such that $M_p \cap f^{-1}(F) = \emptyset$. If $f(p) \in F$, then let $A = (f_{|K})^{-1}(f(p))$. By hypothesis A is finite, therefore there is a compact neighborhood M_p of p in K such that $M_p \cap A = \{p\}$ and $M_p \cap (f^{-1}(F - \{f(p)\})) = \emptyset$.

Given $p \in K$, by construction of M_p, $f_{|M_p}$ is one-to-one. Therefore $f_{|M_p}: M_p \to f(M_p)$ is a homeomorphism. Let L_p be the component of M_p that contains p.

Claim A. *L_p is a neighborhood of p in K.*

Proof. We prove the claim by contradiction. Suppose that L_p is not a neighborhood of p in K, then there are a sequence of points $\{p_n\}_{n=1}^{\infty}$ and a sequence of components $\{L_n\}_{n=1}^{\infty}$ of M_p such that for each $i \neq j$, $L_i \neq L_j$, and for each $n \in \mathbb{N}$, $p_n \in L_n$ and $\lim_{n\to\infty} p_n = p$. Hence $\{f(L_n)\}_{n=1}^{\infty}$ is a family of mutually disjoint subcontinua of the finite graph G. By [6, Theorem 10.4] and since every finite graph is hereditarily locally connected, G contains no convergence continuum. Therefore for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $\operatorname{diam}(f(L_n)) < \varepsilon$ for all $n \geqslant N$. Hence $\lim_{n\to\infty} \operatorname{diam}(f(L_n)) = 0$ implying $\lim_{n\to\infty} \operatorname{diam}(L_n) = 0$. We have then that $\lim_{n\to\infty} L_n = \{p\}$. Since M_p is a neighborhood of p in K, there is an $n \in \mathbb{N}$ such that $L_n \subseteq \operatorname{int}_K(M_p)$. This contradicts the Boundary Bumping Theorem (see [6, Theorem 5.4]). Therefore $p \in \operatorname{int}_K(L_p)$. This proves the claim.

Since L_p is a finite graph, we have that the order of p in K is finite.

If $f(p)$ is not a ramification point of G, we can choose M_p in such a way that $f(M_p)$ has no ramification points of G. Therefore, $f(L_p)$ and L_p are arcs, then the order of p in K is at most 2. Since there is a finite number of ramification points of G and their inverse images under $f_{|K}$ are finite, we have that almost all the points on K have order at most 2. Then Theorem 9.10 of [6] implies that K is a finite graph. □

The following lemma is easy to prove.

3.6. Lemma. *Let $f : X \to G$ be a continuous function and let $K \in C(X)$ be such that $f_{|K}$ is almost one-to-one. Then*:

(a) *If G is an arc, then K is an arc and $f_{|K} : K \to G$ is an embedding.*
(b) *If L is a subcontinuum of K, then $f_{|L} : L \to G$ is almost one-to-one.*
(c) *If K is an Eulerian path over G, then, for each ramification point v of G, $f^{-1}(v)$ is nondegenerate.*

3.7. Lemma. *Let $f : X \to G$ be a Whitney preserving map, $K \in C(X)$ such that K is nondegenerate and let α be an arc in X such that $f_{|K} : K \to f(K)$ is almost one-to-one, $\alpha \cap K = \{p\}$ for some $p \in K$, p is an endpoint of α and $f(\alpha) \subseteq f(K)$. Then K is an arc joining two points u and v, $p \in \{u, v\}$, $f(u) = f(v)$ and $f_{|K}$ is an Eulerian path on $f(K)$.*

Proof. From 3.5, K is a finite graph. Assume f is a μ, ν-Whitney preserving map. Since $f_{|K}$ is almost one-to-one, $f(K)$ is nondegenerate and, by 3.2, f is light. Let $t = \mu(K)$ and $s \in [0, 1)$ be such that $\widehat{f}(\mu^{-1}(t)) = \nu^{-1}(s)$. Therefore $f(K) \in \nu^{-1}(s)$, implying $s > 0$.

We show, by contradiction, that K contains no cycles. Assume that K contains one cycle C. Since $f(C)$ is nondegenerate, there is a point $x_0 \in C$ such that $(f_{|K})^{-1}(f(x_0)) = \{x_0\}$, x_0 is not a ramification point of K and $f(x_0) \neq f(p)$. Let β be a subarc of C such that x_0 is an endpoint of β, β has no ramification points of K, for all $x \in \beta$, $(f_{|K})^{-1}(f(x)) = \{x\}$ and $f(p) \notin f(\beta)$. Note that $\operatorname{cl}_K(K - \beta)$ is a proper subcontinuum of K, therefore $\mu(\operatorname{cl}_K(K - \beta)) < t$. Hence we can choose a subarc γ of α such that $p \in \gamma$, $f(\gamma) \cap f(\beta) = \emptyset$ and $\mu(\operatorname{cl}_K(K - \beta) \cup \gamma) < t$.

Since $\mu(\operatorname{cl}_K(K - \beta) \cup \beta \cup \gamma) > t$, there is a subarc λ of β such that $x_0 \in \lambda$ and $\mu(\operatorname{cl}_K(K - \beta) \cup \lambda \cup \gamma) = t$. Let $A = \operatorname{cl}_K(K - \beta) \cup \lambda \cup \gamma$. Note that $A \subseteq K \cup \gamma$ and that $f(A) \subseteq f(K) \cup f(\gamma) \subseteq f(K) \cup f(\alpha) = f(K)$. Since $f(A), f(K) \in \nu^{-1}(s)$, we have that $f(A) = f(K)$.

Given that $A \neq \operatorname{cl}_K(K - \beta) \cup \beta \cup \gamma$, there is a point $u \in \beta - A$ and, by the way β was chosen, $(f_{|K})^{-1}(f(u)) = \{u\}$. Also $f(u) \in f(\beta)$, so $f(u) \notin f(\gamma)$. In addition, $f(u) \notin f(A)$ because $A \subseteq K \cup \gamma$ and $u \notin A$. Therefore $f(u) \in f(K) - f(A)$, which is a contradiction. This shows that K has no cycles.

Hence K is a tree.

Since $f_{|K}$ is almost one-to-one, the number $\varepsilon = \min\{\operatorname{diam}(f(L))\colon L \text{ is an edge of } K\}$ is positive. By 3.2, f is a light map. Let η be a subarc of α such that $p \in \eta$, $\operatorname{diam}(f(\eta)) < \varepsilon$, and $\operatorname{diam}(\eta) < \delta$, where δ is as in 3.1.

Claim B. *If $e \neq p$ is an end point of K and J is the edge of K containing e, then $f(e) = f(p)$ and there exists a subarc J_0 of J such that $e \in J_0$ and $f(J_0) \subseteq f(\eta)$.*

Proof. Let v be the other end point of J. Let $\sigma : [0, 1] \to J$ be a homeomorphism such that $\sigma(0) = v$ and $\sigma(1) = e$. Let $K_0 = (K - J) \cup \{v\}$. Note that K_0 is a proper subcontinuum of K. Since $p \neq e$, there exists a subcontinuum K_1 of K such that $K_0 \cup \{p\} \subseteq K_1$ and $e \notin K_1$. Given that $\mu(K_1) < t$, there exists a subarc η_1 of η such that $p \in \eta_1$ and $\mu(K_1 \cup \eta_1) < t$.

Let η_0 be any subarc of η_1 with the property that $p \in \eta_0$. For each $r \in [0, 1]$, let $L_r = K_1 \cup \eta_0 \cup \sigma([0, r])$. Note that $\mu(L_0) < t$ and $\mu(L_1) > t$, therefore there exists $r \in (0, 1)$ such that $\mu(L_r) = t$. Notice that $e \notin L_r$. By the way s was chosen, $f(L_r) \in \nu^{-1}(s)$. Since $f(L_r) \subseteq f(L_1) = f(K) \cup f(\eta_0) = f(K)$ and $\nu(f(L_r)) = \nu(f(K))$, we have $f(L_r) = f(K)$.

Given $x \in J - (\sigma([0, r]) \cup K_1)$ and any subarc M of $J - (\sigma([0, r]) \cup K_1)$ containing x, since $f(M)$ is nondegenerate and $f_{|K}$ is almost one-to-one, there exists $u \in M$ such that $\{u\} = (f_{|K})^{-1}(f(u))$. Since $f(u) \in f(K) = f(L_r) = f(K_1) \cup f(\sigma([0, r])) \cup f(\eta_0)$ and $u \notin \sigma([0, r]) \cup K_1 \subseteq K$, we have that $f(u) \in f(\eta_0)$. Given that M is any subarc, $f(x) \in f(\eta_0)$. We have proved that $f(J - (\sigma([0, r]) \cup K_1)) \subseteq f(\eta_0)$. In particular, $f(e) \in f(\eta_0)$ and, since η_0 was arbitrarily chosen, $f(e) = f(p)$.

Now, let $\eta_0 = \eta_1$ and let r be as given above, then $J_0 = \operatorname{cl}_X(J - (\sigma([0, r]) \cup K_1))$ is a subarc of J, $e \in J_0$, and $f(J_0) \subseteq f(\eta)$. This proves the claim.

Recall that K is a tree.

Let e_1 and e_2 be two different end points of K. We show, by contradiction, that $p \in \{e_1, e_2\}$. So, assume $p \notin \{e_1, e_2\}$. Let J_1 and J_2 be the edges of K that contain e_1 and e_2, respectively. (J_1 and J_2 may be the same, in the case that K is an arc.) By the claim, $f(e_1) = f(e_2) = f(p)$ and there are two subarcs P_1 and P_2 of J_1 and J_2, respectively, such that $e_1 \in P_1$, $e_2 \in P_2$ and $f(P_1) \cup f(P_2) \subseteq f(\eta)$. We can also assume that $P_1 \cap P_2 = \emptyset$.

By the choice of δ and η, and since f is light, 3.1 implies $f_{|\eta} : \eta \to f(\eta)$ is a homeomorphism. Hence, since p is an end point of η, $f(p)$ is an end point of $f(\eta)$. We can assume that $f(P_1) \subseteq f(P_2)$, due to the facts that $f(P_1)$ and $f(P_2)$ are subcontinua of $f(\eta)$, and that $f(p)$ is contained in both. Note that $f(P_1)$ is nondegenerate and that, for each $y \in f(P_1)$, $(f_{|K})^{-1}(y)$ is nondegenerate, since it intersects both P_1 and P_2. This is a contradiction to $f_{|K}$ being almost one-to-one. Therefore $p \in \{e_1, e_2\}$.

The previous argument also shows that K must be an arc. Hence p is an end point of K. Let u be the other end point of K. By the claim, $f(u) = f(p)$. $\square$

3.8. Lemma. *Let X be a continuum and let $f : X \to G$ be a continuous light function. Let $\{K_n\}_{n=1}^{\infty}$ be a collection of subcontinua of X such that $K_1 \subseteq K_2 \subseteq \cdots$ and $f_{|K_n} : K_n \to f(K_n)$ is almost one-to-one, for all $n \in \mathbb{N}$. Let $K = \mathrm{cl}_X(\bigcup\{K_n\colon n \in \mathbb{N}\})$. Then $f_{|K} : K \to f(K)$ is almost one-to-one.*

Proof. Let $q \in G$ be any point of G. Assume that the order of q in G is m. We prove by contradiction that the set $D = (f_{|K})^{-1}(q)$ has at most m points. So, suppose that D has at least $m+1$ points. Since f is light, D has at least $m+1$ components. Hence there are $m+1$ mutually disjoint, nonempty compact sets $D_1, \ldots, D_{m+1}$ such that $D = D_1 \cup \cdots \cup D_{m+1}$. Let $U_1, \ldots, U_{m+1}$ be open sets of X such that, for each $i \in \{1, \ldots, m+1\}$, $D_i \subseteq U_i$ and the sets $\mathrm{cl}_X(U_1), \ldots, \mathrm{cl}_X(U_{m+1})$ are mutually disjoint. Let $U = U_1 \cup \cdots \cup U_{m+1}$. Since the set $M = f(K - U)$ is compact and $q \notin M$, there exists a simple m-od R such that R is a neighborhood of q in G and $R \cap M = \emptyset$. We let $R = L_1 \cup \cdots \cup L_m$, where each L_i is an arc having q as an end point, and $L_i \cap L_j = \{q\}$, if $i \neq j$. For each $i \in \{1, \ldots, m\}$, let q_i denote the end point of L_i which is different from q. We can assume that $R - \{q_1, \ldots, q_m\}$ is an open set of G. Note that $(f_{|K})^{-1}(R) \subseteq U$.

For each $i \in \{1, \ldots, m+1\}$, choose a point $p_i \in D_i$ and let $W_i = U_i \cap f^{-1}(R - \{q_1, \ldots, q_m\})$. Then W_i is an open set of X with $p_i \in W_i$. Since $p_i \in K$, there exists $N_i \in \mathbb{N}$ such that $W_i \cap K_{N_i} \neq \emptyset$. Let $N = \max\{N_1, \ldots, N_{m+1}\}$. We have that $W_i \cap K_N \neq \emptyset$ for all i. Since K_N is connected and intersects each of the open, mutually disjoint, sets $W_1, \ldots, W_{m+1}$, we have that K_N is not contained in the union $W_1 \cup \cdots \cup W_{m+1}$. Furthermore, since K_N is arcwise connected, for each $i \in \{1, \ldots, m+1\}$, there exists a one-to-one continuous function $\alpha_i : [0,1] \to K_N$ such that $\alpha_i(0) = p_i$, $\alpha_i(1) \notin W_i$ and $\alpha_i([0,1)) \subseteq W_i$. We claim that $f(\alpha_i(1)) \in \{q_1, \ldots, q_m\}$. Assume, to the contrary, that $f(\alpha_i(1)) \notin \{q_1, \ldots, q_m\}$. Since $f(W_i) \subseteq R$, we have that $f(\alpha_i(1)) \in R - \{q_1, \ldots, q_m\}$. Given that $\alpha_i(1) \notin W_i$, we have $\alpha_i(1) \notin U_i$, but $\alpha_i(1) \in \mathrm{cl}_X(U_i)$. Therefore $\alpha_i(1) \notin U$. This is a contradiction since $\alpha_i(1) \in (f_{|K})^{-1}(R) \subseteq U$. This shows that $f(\alpha_i(1)) \in \{q_1, \ldots, q_m\}$. By the pigeonhole principle, we can assume that $f(\alpha_1(1)) = f(\alpha_2(1)) = q_1$. Then $f(\alpha_1([0,1]))$ and $f(\alpha_2([0,1]))$ are two nondegenerate (f is light) subcontinua of the simple n-od R, and both contain the point q_1. Hence these subcontinua must intersect in, at least, an arc L. Given $y \in L$, $(f_{|K_N})^{-1}(y)$ intersects $\mathrm{cl}_X(W_1)$ and $\mathrm{cl}_X(W_2)$, which are disjoint; hence $(f_{|K_N})^{-1}(y)$ is nondegenerate. This is a contradiction to $f_{|K_N}$ being almost one-to-one. This shows that $(f_{|K})^{-1}(q)$ has at most m points.

In particular, for each $q \in G$, $(f_{|K})^{-1}(q)$ is finite.

Let J be an edge of G with end points y and v. Let $J_1 = J - \{y, v\}$. We show that there is, at most, one point $w \in J_1$ such that $(f_{|K})^{-1}(w)$ is nondegenerate. Assume, to the contrary, that there are two different points $w_1, w_2 \in J_1$ such that $(f_{|K})^{-1}(w_1)$ and $(f_{|K})^{-1}(w_2)$ are nondegenerate. Define a linear order $<$ on J and assume that $y < w_1 < w_2 < v$ holds. Let W_1 and W_2 be compact neighborhoods of w_1 and w_2, respectively, in G such that $W_1 \cap W_2 = \emptyset$ and $W_1 \cup W_2 \subseteq J_1$. From what we proved earlier, each of the sets $(f_{|K})^{-1}(w_1)$ and $(f_{|K})^{-1}(w_2)$ has two elements. Assume that $(f_{|K})^{-1}(w_1) = \{a_1, a_2\}$ and $(f_{|K})^{-1}(w_2) = \{b_1, b_2\}$. Let A_1, A_2, B_1 and B_2 be mutually disjoint open sets of X with $a_1 \in A_1$, $a_2 \in A_2$, $b_1 \in B_1$ and $b_2 \in B_2$, and such that $A_1 \cup A_2 \subseteq f^{-1}(W_1)$ and $B_1 \cup B_2 \subseteq f^{-1}(W_2)$.

Since $f(K - (A_1 \cup A_2))$ is a compact subset of G that does not contain w_1, we can pick points s_1 and s_2 in J_1 such that $y < s_1 < w_1 < s_2 < w_2 < v$, and the subinterval $[s_1, s_2]$ of J_1 and $f(K - (A_1 \cup A_2))$ are disjoint. Note that $(f_{|K})^{-1}([s_1, s_2]) \subseteq A_1 \cup A_2$. Similarly, we can pick points t_1 and t_2 in J_1 such that $y < s_1 < w_1 < s_2 < t_1 < w_2 < t_2 < v$, the interval $[t_1, t_2]$ of J_1 and $f(K - (B_1 \cup B_2))$ are disjoint, and such that $(f_{|K})^{-1}([t_1, t_2]) \subseteq B_1 \cup B_2$. Let $P_1 = A_1 \cap f^{-1}((s_1, s_2))$, $P_2 = A_2 \cap f^{-1}((s_1, s_2))$, $Q_1 = B_1 \cap f^{-1}((t_1, t_2))$ and $Q_2 = B_2 \cap f^{-1}((t_1, t_2))$. Then P_1, P_2, Q_1 and Q_2 are open sets of X and contain, respectively, the points a_1, a_2, b_1 and b_2. Since these points belong to K, there is an $N \in \mathbb{N}$ such that K_N intersects each of the sets P_1, P_2, Q_1 and Q_2. Select points $p_1 \in P_1 \cap K_N$, $p_2 \in P_2 \cap K_N$, $q_1 \in Q_1 \cap K_N$ and $q_2 \in Q_2 \cap K_N$. We can assume that $f(p_1) \leqslant f(p_2)$.

If there is a subarc R of K_N such that $p_1, p_2 \in R$ and $f(R) \subseteq J$, by 3.6, $f_{|R} : R \to J$ is an embedding. Let $L = (f_{|R})^{-1}([f(p_1), f(p_2)])$, then L is a subcontinuum of R containing the points p_1 and p_2; furthermore $L \subseteq f^{-1}((s_1, s_2))$. Hence $L \subseteq (f_{|R})^{-1}([s_1, s_2]) \subseteq (f_{|K})^{-1}([s_1, s_2]) \subseteq A_1 \cup A_2$. However L is connected, $p_1 \in L \cap A_1$ and $p_2 \in L \cap A_2$. Since A_1 and A_2 are open disjoint subsets of X, we have a contradiction to the connectedness of L. This shows that such an R does not exist. In particular, it also shows that $f(K_N)$ is not contained in J. Now, choose a point $z \in K_N$ such that $f(z) \notin J$. In a similar way, we can prove that there is no subarc S in K_N such that $q_1, q_2 \in S$ and $f(S) \subseteq J$.

Consider arcs in K_N joining p_1 and p_2 with z, we can find one-to-one continuous functions α_1 and α_2 from $[0,1]$ to K_N such that $\alpha_1(0) = p_1$, $\alpha_2(0) = p_2$, $\alpha_1([0,1)) \cup \alpha_2([0,1)) \subseteq f^{-1}(J_1)$ and $\{\alpha_1(1), \alpha_2(1)\} \cap f^{-1}(J_1) = \emptyset$. Note that $\{f(\alpha_1(1)), f(\alpha_2(1))\} \subseteq \{y, v\}$. From the previous paragraph we have that $p_2 \notin \mathrm{Im}(\alpha_1)$ and $p_1 \notin \mathrm{Im}(\alpha_2)$. By 3.6, $f_{|\mathrm{Im}(\alpha_1)}: \mathrm{Im}(\alpha_1) \to J$ and $f_{|\mathrm{Im}(\alpha_2)}: \mathrm{Im}(\alpha_2) \to J$ are embeddings. Suppose that $f(\alpha_1(1)) = f(\alpha_2(1))$, and assume that $f(\alpha_1(1)) = v$. Then, for each $i \in \{1,2\}$, $f(\mathrm{Im}(\alpha_i)) = [f(p_i), v]$. Let λ be a subarc of $\mathrm{Im}(\alpha_2)$ such that $p_2 \in \lambda$ and $\lambda \cap \mathrm{Im}(\alpha_1) = \emptyset$. Then $f(\lambda) \subseteq [f(p_1), v] = f(\mathrm{Im}(\alpha_1))$. This is impossible since $f_{|K_N}$ is almost one-to-one. This shows that $f(\alpha_1(1)) \neq f(\alpha_2(1))$. Therefore $\{f(\alpha_1(1)), f(\alpha_2(1))\} = \{y, v\}$.

Assume $f(\alpha_1(1)) = v$. We have that $f(\mathrm{Im}(\alpha_1)) = [f(p_1), v]$. From what we proved two paragraphs above, q_1 and q_2 cannot both belong to $\mathrm{Im}(\alpha_1)$. Assume $q_1 \notin \mathrm{Im}(\alpha_1)$. Let η be a subarc of K_N such that $q_1 \in \eta$ and $\eta \cap \mathrm{Im}(\alpha_1) = \emptyset$. Since $f(q_1) \in (f(p_1), v)$, we can assume that $f(\eta) \subseteq (f(p_1), v)$. But then $f(\eta) \subseteq f(\mathrm{Im}(\alpha_1))$ and $\eta \cap \mathrm{Im}(\alpha_1) = \emptyset$. This contradicts the fact that $f_{|K_N}$ is almost one-to-one; showing that there is at most one point $w \in J_1$ such that $(f_{|K})^{-1}(w)$ is nondegenerate.

We have showed that the set $\{g \in G\colon (f_{|K})^{-1}(g)$ is nondegenerate$\}$ consists of, at most, the ramification points of G, the end points of G and one more point from each edge of G. Therefore this set is finite. Hence $f_{|K}: K \to G$ is almost one-to-one. $\square$

3.9. Lemma. *Let $f: X \to G$ be a Whitney preserving map. Assume that X has an arcwise connected dense subset E. Let $K \in C(X)$ such that K is nondegenerate, $K \subseteq E$ and such that $f_{|K}$ is almost one-to-one. Suppose that $f(K) \neq G$. Then one of the following statements holds*:

(a) *There exists $K_1 \in C(X)$ such that $K \subseteq K_1$, $f(K_1) \neq f(K)$ and $f_{|K_1}$ is almost one-to-one.*
(b) *K is an arc, $f_{|K}$ is an Eulerian path on $f(K)$ and there is $K_1 \in C(X)$ such that $f_{|K_1}$ is almost one-to-one and $f(K) \subseteq f(K_1) \neq f(K)$.*

Proof. By 3.3, f is light. Assume that f is μ, ν-Whitney preserving. Let $t = \mu(K)$ and let $s \in [0,1]$ be such that $\widehat{f}(\mu^{-1}(t)) = \nu^{-1}(s)$. Then $f(K) \in \nu^{-1}(s)$. Since f is light, $s > 0$. Let α be an arc in E and $p \in K$ be such that $K \cap \alpha = \{p\}$, p is an end point of α and $f(\alpha) - f(K) \neq \emptyset$. Let p_0 be the end point of α that is different from p, we assume that $f(p_0) \notin f(K)$. Let $2\varepsilon = \min\{\mathrm{diam}(J)\colon J$ is an edge of $G\}$. For $\varepsilon > 0$, let $\delta > 0$ be given by ε and the uniform continuity of f, and such that it satisfies the conditions of 3.1.

Take a subarc η of X such that $\mathrm{diam}(\eta) < \delta$, then $\mathrm{diam}(f(\eta)) < \varepsilon$. Since K is a finite graph (by 3.5), we can write $K = \beta_1 \cup \cdots \cup \beta_m$, where $m \in \mathbb{N}$ and, for each $i \in \{1, \ldots, m\}$, β_i is an arc and $\mathrm{diam}(\beta_i) < \delta$, hence $\mathrm{diam}(f(\beta_i)) < \varepsilon$. Let $i \in \{1, \ldots, m\}$. In the case that $f(\eta) \cap f(\beta_i) \neq \emptyset$, by the way δ was chosen, $f(\eta)$ and $f(\beta_i)$ are arcs. If $f(\eta) \cap f(\beta_i)$ is disconnected, then $f(\eta) \cup f(\beta_i)$ contains a simple closed curve. This is impossible since $\mathrm{diam}(f(\eta) \cup f(\beta_i)) < 2\varepsilon$. This shows that $f(\eta) \cap f(\beta_i)$ is connected. Therefore $f(\eta) \cap f(K) = f(\eta) \cap (f(\beta_1) \cup \cdots \cup f(\beta_m))$ has a finite number of components. We have showed that, if η is a subarc of X such that $\mathrm{diam}(\eta) < \delta$, then $f(\eta) \cap f(K)$ has a finite number of components. Since every arc in X can be divided into subarcs with diameter is less than δ, we have that, for each subarc η of X, $f(\eta) \cap f(K)$ has a finite number of components.

We consider two cases.

Case 1. There is not a subarc η of α such that $p \in \eta$ and $f(\eta) \subseteq f(K)$.

Let η be a subarc of α such that $p \in \eta$ and $\mathrm{diam}(\eta) < \delta$. From what we proved above, $f(\eta) \cap f(K)$ has a finite number of components. Let L be the component of $f(\eta) \cap f(K)$ that contains $f(p)$. If L is nondegenerate, since $f_{|\eta}: \eta \to f(\eta)$ is a homeomorphism, there exists a subarc η_0 of η such that $p \in \eta_0$ and $f(\eta_0) = L \subseteq f(K)$; however this contradicts the hypothesis of Case 1. Hence $L = \{f(p)\}$. Given that $f(\eta) \cap f(K)$ has a finite number of components and it is a subset of the arc $f(\eta)$, there is a subarc L_1 of $f(\eta)$ such that $L_1 \cap f(\eta) \cap f(K) = \{f(p)\}$. Let η_1 be a subarc of η such that $p \in \eta_1$ and $f(\eta_1) = L_1$. Let $K_1 = K \cup \eta_1$. Then K_1 is a finite graph and $K \subseteq K_1$. Since $L_1 \cap f(K) = \{f(p)\}$, $f(K) \neq f(K_1)$. Clearly, $f_{|K_1}$ is almost one-to-one. Therefore, in this case, (a) holds.

Case 2. There is a subarc η of α such that $p \in \eta$ and $f(\eta) \subseteq f(K)$.

By 3.7, there exists a point $u \in K - \{p\}$ such that K is an arc joining u with p, $f(u) = f(p)$ and $f_{|K}$ is an Eulerian path on $f(K)$. Since $K \cup \alpha$ is an arc, there is a homeomorphism $\sigma: [0,2] \to K \cup \alpha$ such that $\sigma(0) = u$, $\sigma(1) = p$, $\sigma([0,1]) = K$ and $\sigma([1,2]) = \alpha$. Let $r_0 = \max\{r \in [1,2]\colon f(\sigma([1,r])) \subseteq f(K)\}$. By the hypothesis, for this case, $1 < r_0$. Given $r \in [1, r_0]$, since $K \subseteq \sigma([0,r])$, there exists a number $\varphi(r) \in [0,r)$ such that $\mu(\sigma([\varphi(r), r])) = t$. Let $A(r) = \sigma([\varphi(r), r])$. Note that $\varphi(r)$ is unique, $\varphi(1) = 0$, and that the map φ is strictly increasing and it depends continuously on r.

Since $f(\sigma([1, r_0])) \subseteq f(K)$ and $f(\sigma([0,1])) = f(K)$, we have that $f(\sigma([0, r_0])) = f(K)$. Let $r \in [1, r_0]$. Given that $A(r) \subseteq \sigma([0, r_0])$, $f(A(r)) \subseteq f(K)$. By the choice of r, $A(r) \in \mu^{-1}(t)$, hence $\nu(f(A(r))) = s$ and since $\nu(f(K)) = s$, we have that $f(A(r)) = f(K)$.

Let $\lambda > 0$ be such that it satisfies the following implication: if $r, u \in [1, r_0]$ and $r \leqslant u \leqslant r + \lambda$, then $\mathrm{diam}(\sigma([r,u])) < \delta$, $\mathrm{diam}(\sigma([\varphi(r), \varphi(u)])) < \delta$ and $\mu(\sigma([r,u])) < t$. Let $P = \{r_1, \ldots, r_m\}$ be a partition of $[1, r_0]$ such that $1 = r_1 < \cdots < r_m = r_0$ and such that for each $i \in \{1, \ldots, m-1\}$, $|r_{i+1} - r_i| < \lambda$. Given $i \in \{1, \ldots, m-1\}$, since $\mu(\sigma([r_i, r_{i+1}])) < t$, we have that $\varphi(r_{i+1}) < r_i$. We prove the following claim.

Claim C. *Let $i \in \{1, \ldots, m-1\}$. Assume that $f_{|A(r_i)}$ is almost one-to-one and that $f(\sigma(\varphi(r_i))) = f(\sigma(r_i))$. Then $f(\sigma(\varphi(r_{i+1}))) = f(\sigma(r_{i+1}))$ and $f(\sigma([\varphi(r_i), \varphi(r_{i+1})])) = f(\sigma([r_i, r_{i+1}]))$.*

Proof. By the choice of δ, $f_{|\sigma([\varphi(r_i),\varphi(r_{i+1})])}$ and $f_{|\sigma([r_i,r_{i+1}])}$ are one-to-one, therefore $f(\sigma([\varphi(r_i), \varphi(r_{i+1})]))$ and $f(\sigma([r_i, r_{i+1}]))$ are two arcs with the point $f(\sigma(\varphi(r_i))) = f(\sigma(r_i))$ in common, which is an end point of each arc. Since the union of these arcs has diameter less than 2ε, the union contains no simple closed curves, hence the intersection of these arcs is connected. Therefore there exist $u_1 \in [\varphi(r_i), \varphi(r_{i+1})]$ and $u_2 \in [r_i, r_{i+1}]$ such that

$$f\big(\sigma\big(\big[\varphi(r_i), \varphi(r_{i+1})\big]\big)\big) \cap f\big(\sigma\big([r_i, r_{i+1}]\big)\big) = f\big(\sigma\big(\big[\varphi(r_i), u_1\big]\big)\big) = f\big(\sigma\big([r_i, u_2]\big)\big)$$

and $f(\sigma(u_1)) = f(\sigma(u_2))$.

We show that $f(\sigma([\varphi(r_{i+1}), u_2])) = f(K)$. Note that $f(\sigma([\varphi(r_{i+1}), u_2])) \subseteq f(K)$, thus we just need to show that the other inclusion holds. Given that $f(\sigma([\varphi(r_i), r_i])) = f(A(r)) = f(K) = f(\sigma([\varphi(r_{i+1}), r_{i+1}]))$,

$$f\big(\sigma\big(\big(u_1, \varphi(r_{i+1})\big]\big)\big) \subseteq f(K) = f\big(\sigma\big(\big[\varphi(r_{i+1}), r_{i+1}\big]\big)\big).$$

By our choice of u_1 and u_2, $f(\sigma((u_1, \varphi(r_{i+1})])) \cap f(\sigma((u_2, r_{i+1}])) = \emptyset$, thus

$$f\big(\sigma\big(\big[u_1, \varphi(r_{i+1})\big]\big)\big) \subseteq f\big(\sigma\big(\big[\varphi(r_{i+1}), u_2\big]\big)\big).$$

Hence

$$\begin{aligned} f(K) = f\big(A(r_i)\big) &= f\big(\sigma\big(\big[\varphi(r_i), r_i\big]\big)\big) \\ &= f\big(\sigma\big(\big[\varphi(r_i), u_1\big]\big)\big) \cup f\big(\sigma\big(\big[u_1, \varphi(r_{i+1})\big]\big)\big) \cup f\big(\sigma\big(\big[\varphi(r_{i+1}), r_i\big]\big)\big) \\ &= f\big(\sigma\big([r_i, u_2]\big)\big) \cup f\big(\sigma\big(\big[u_1, \varphi(r_{i+1})\big]\big)\big) \cup f\big(\sigma\big(\big[\varphi(r_{i+1}), r_i\big]\big)\big) \\ &\subseteq f\big(\sigma\big(\big[\varphi(r_{i+1}), u_2\big]\big)\big) \cup f\big(\sigma\big([r_i, u_2]\big)\big) \cup f\big(\sigma\big(\big[\varphi(r_{i+1}), r_i\big]\big)\big) = f\big(\sigma\big(\big[\varphi(r_{i+1}), u_2\big]\big)\big). \end{aligned}$$

This proves $f(\sigma([\varphi(r_{i+1}), u_2])) = f(K)$.

Next, we show that $u_2 = r_{i+1}$. Assume, to the contrary, that $u_2 < r_{i+1}$. Then $\sigma([\varphi(r_{i+1}), u_2])$ is a proper subcontinuum of $\sigma([\varphi(r_{i+1}), r_{i+1}])$. Therefore $\mu(\sigma([\varphi(r_{i+1}), u_2])) < \mu(\sigma([\varphi(r_{i+1}), r_{i+1}])) = t$. Let $t_1 = \mu(\sigma([\varphi(r_{i+1}), u_2]))$. Because f is μ, ν-Whitney preserving and $f(\sigma([\varphi(r_{i+1}), u_2])) = f(K) \in \nu^{-1}(s)$, we have that $f(\mu^{-1}(t_1)) = \nu^{-1}(s)$. Using an order arc, we can construct a subcontinuum K_0 of K such that $\mu(K_0) = t_1$. Hence K_0 is a proper subcontinuum of K. Since $f(K_0) \subseteq f(K)$ and $\nu(f(K_0)) = s = \nu(f(K))$, we have that

$$f(K_0) = f(K). \tag{3.9.1}$$

Using order arcs, we can construct a nondegenerate subcontinuum B of K such that $B \cap K_0 = \emptyset$. Hence, from (3.9.1), $f(B) \subseteq f(K_0)$. This implies that $f_{|K}$ is not one-to-one at any point of B, contradicting the fact that $f_{|K}$ is almost one-to-one. The contradiction shows that $u_2 = r_{i+1}$.

Now, we show that $u_1 = \varphi(r_{i+1})$. Assume, to the contrary, that $u_1 < \varphi(r_{i+1})$. Since $f_{|\sigma([\varphi(r_i),r_i])}$ is almost one-to-one and $(u_1, \varphi(r_{i+1})) \subseteq [\varphi(r_i), r_i]$, there exists $u_0 \in (u_1, \varphi(r_{i+1}))$ such that $\{\sigma(u_0)\} = (f_{|\sigma([\varphi(r_i),r_i])})^{-1}(f(\sigma(u_0)))$. Since $f(\sigma(u_0)) \in f(K) = f(\sigma([\varphi(r_{i+1}), r_{i+1}]))$, there exists $v \in [\varphi(r_{i+1}), r_{i+1}]$ such that $f(\sigma(u_0)) = f(\sigma(v))$. If $v \leqslant r_i$, then $v \in [\varphi(r_i), r_i]$. By our choice of u_0, $v = u_0$, which is impossible since $u_0 < \varphi(r_{i+1}) \leqslant v$. This shows that $r_i < v$. Therefore $f(\sigma(u_0)) \in f(\sigma([u_1, \varphi(r_{i+1})])) \cap f(\sigma([r_i, r_{i+1}])) \subseteq f(\sigma([\varphi(r_i), \varphi(r_{i+1})])) \cap f(\sigma([r_i, r_{i+1}])) = f(\sigma([\varphi(r_i), u_1]))$. Then there exists $w \in [\varphi(r_i), u_1] \subseteq [\varphi(r_i), r_i]$ such that $f(\sigma(u_0)) = f(\sigma(w))$. By the choice of u_0, $u_0 = w$. This is also impossible since $w \leqslant u_1 < u_0$. This proves that $u_1 = \varphi(r_{i+1})$.

From the definitions of u_1 and u_2, $f(\sigma([\varphi(r_i), \varphi(r_{i+1})])) = f(\sigma([\varphi(r_i), u_1])) = f(\sigma([r_i, u_2])) = f(\sigma([r_i, r_{i+1}]))$. Since $(f \circ \sigma)_{|[\varphi(r_i),\varphi(r_{i+1})]}$ and $(f \circ \sigma)_{|[r_i,r_{i+1}]}$ are one-to-one and, by hypothesis, $f(\sigma(\varphi(r_i))) = f(\sigma(r_i))$, we have that $f(\sigma(\varphi(r_{i+1}))) = f(\sigma(r_{i+1}))$. This ends the proof of Claim C.

Claim D. *Let $i \in \{1, \ldots, m\}$. Then $f_{|A(r_i)}$ is almost one-to-one and $f(\sigma(\varphi(r_i))) = f(\sigma(r_i))$.*

Proof. We prove the claim by induction. If $i = 1$, note that $A(r_1) = \sigma([\varphi(r_1), r_1]) = \sigma([0, 1]) = K$. Hence $f_{|A(r_1)}$ is almost one-to-one and $f(\sigma(\varphi(r_1))) = f(u) = f(p) = f(\sigma(r_1))$.

Suppose that $i \in \{1, \ldots, m-1\}$ and that Claim D holds for i. We show that it also holds for $i+1$. By Claim C, $f(\sigma(\varphi(r_{i+1}))) = f(\sigma(r_{i+1}))$ and $f(\sigma([\varphi(r_i), \varphi(r_{i+1})])) = f(\sigma([r_i, r_{i+1}]))$.

It just remains to show that $f_{|\sigma([\varphi(r_{i+1}),r_{i+1}])}$ is almost one-to-one. Let $y \in f(\sigma([\varphi(r_{i+1}), r_{i+1}])) = f(K) = f(\sigma([\varphi(r_i), r_i]))$. Let us assume first that $y \notin \{f(\sigma(r_i)), f(\sigma(r_{i+1}))\}$. Recall that $f_{|\sigma([\varphi(r_i),r_i])}$ is almost one-to-one. We show that the sets

$$f^{-1}(y) \cap \sigma\big(\big[\varphi(r_i), r_i\big]\big) = \big(f^{-1}(y) \cap \sigma\big(\big[\varphi(r_i), \varphi(r_{i+1})\big)\big)\big) \cup \big(f^{-1}(y) \cap \sigma\big(\big[\varphi(r_{i+1}), r_i\big]\big)\big)$$

and

$$f^{-1}(y) \cap \sigma\big(\big[\varphi(r_{i+1}), r_{i+1}\big]\big) = \big(f^{-1}(y) \cap \sigma\big(\big[\varphi(r_{i+1}), r_i\big]\big)\big) \cup \big(f^{-1}(y) \cap \sigma\big((r_i, r_{i+1}]\big)\big)$$

have the same number of elements. To do this, it is enough to show that

$$f^{-1}(y) \cap \sigma\big(\big[\varphi(r_i), \varphi(r_{i+1})\big)\big) \quad \text{and} \quad f^{-1}(y) \cap \sigma\big(\big(r_i, r_{i+1}\big]\big)$$

have the same number of elements. If there is $x \in f^{-1}(y) \cap \sigma([\varphi(r_i), \varphi(r_{i+1})))$, since $f(\sigma(\varphi(r_i))) = f(\sigma(r_i))$, we have that $x \neq \sigma(\varphi(r_i))$. Because $f_{|\sigma([\varphi(r_i),\varphi(r_{i+1})])}$ and $f_{|\sigma([r_i,r_{i+1}])}$ are one-to-one and $f(\sigma((\varphi(r_i), \varphi(r_{i+1})))) = f(\sigma((r_i, r_{i+1})))$, we have that x is unique and there exists a unique element $w \in f^{-1}(y) \cap \sigma((r_i, r_{i+1}])$ such that $f(w) = f(x)$. This shows that if $f^{-1}(y) \cap \sigma([\varphi(r_i), \varphi(r_{i+1}))) \neq \emptyset$, then this set consists of one element and the same is true for $f^{-1}(y) \cap \sigma((r_i, r_{i+1}])$. Similarly, we can show that if $f^{-1}(y) \cap \sigma((r_i, r_{i+1}]) \neq \emptyset$, then the sets $f^{-1}(y) \cap \sigma((r_i, r_{i+1}])$ and $f^{-1}(y) \cap \sigma([\varphi(r_i), \varphi(r_{i+1})))$ have exactly one element. Therefore $f^{-1}(y) \cap \sigma((r_i, r_{i+1}])$ and $f^{-1}(y) \cap \sigma([\varphi(r_i), \varphi(r_{i+1})))$ have the same number of elements. The case in which $y = f(\sigma(r_i))$, $f^{-1}(y) \cap \sigma([\varphi(r_i), \varphi(r_{i+1})))$ consists of one element while $f^{-1}(y) \cap \sigma((r_i, r_{i+1}]) = \emptyset$. Thus, for $y = f(\sigma(r_i))$, the difference between the number of elements in the sets $f^{-1}(y) \cap \sigma([\varphi(r_i), \varphi(r_{i+1})))$ and $f^{-1}(y) \cap \sigma((r_i, r_{i+1}])$ is at most one. Something similar holds for the case in which $y = f(\sigma(r_{i+1}))$. This ends the proof that $f_{|\sigma([\varphi(r_{i+1}), r_{i+1}])}$ is almost one-to-one; therefore Claim D is true.

Let $K_2 = A(r_m)$. By Claim D, K_2 is an arc in X such that $f(K_2) = f(K)$ and $f_{|K_2}$ is an Eulerian path on $f(K)$. Since $f(p_0) \notin f(K)$, $p_0 \neq \sigma(r_0)$. Denote be α_1 the subarc of α that joins $\sigma(r_0)$ with p_0. Then $\alpha_1 \cap K_2 = \{\sigma(r_0)\}$. By the definition of r_0, no subarc η_0 of α_1 containing $\sigma(r_0)$ can be contained in $f(K) = f(K_2)$. Then K_2 and α_1 meet the conditions from Case 1; thus we can obtain the conclusion (a) for these particular sets. That is, there exists $K_1 \in C(X)$ such that $K_2 \subseteq K_1$, $f(K_2) \neq f(K_1)$ and $f_{|K_1}$ is almost one-to-one. Then $f(K) = f(K_2) \subseteq f(K_1) \neq f(K_2)$. □

3.10. Theorem. *Let $f : X \to G$ be a Whitney preserving function. Assume X has an arcwise connected dense component E and that G is nondegenerate. Then there exists $K \in C(X)$ such that $K \subseteq E$, $f_{|K}$ is almost one-to-one and $f(K) = G$.*

Proof. By 3.3, f is a light map. Order the set

$$\mathcal{K} = \big\{K \in C(X)\colon\ K \subseteq E \text{ and } f_{|K} \text{ is almost one-to-one}\big\}$$

with the inclusion. We use the Brouwer Reduction Theorem to show that $\mathcal{K}$ has maximal elements. By 3.1, f restricted to any sufficiently small arc is one-to-one. Hence $\mathcal{K}$ is nonempty. By 3.8, every increasing sequence of elements of $\mathcal{K}$ has an upper bound in $\mathcal{K}$. Then, by the Brouwer Reduction Theorem, every element of $\mathcal{K}$ is contained in a maximal element.

Assume f is μ, ν-Whitney preserving.

Let

$$\mathcal{G} = \big\{H \in C(G)\colon\ \text{there is } K \in C(X) \text{ such that } K \subseteq E,\ f_{|K} \text{ is almost one-to-one and } f(K) = H\big\}.$$

Since $\mathcal{K}$ is nonempty, $\mathcal{G}$ is nonempty. Let $t_0 = \sup\{\nu(H) \in [0, \nu(G)]\colon\ H \in \mathcal{G}\}$. Then there is a sequence, $\{K_n\}_{n=1}^{\infty}$, of elements of $\mathcal{K}$ such that $\lim_{n\to\infty} \nu(f(K_n)) = t_0$. We can assume that $\lim_{n\to\infty} f(K_n) = G_0$ for some $G_0 \in C(G)$. Then $\nu(G_0) = t_0$. From the previous argument, we can assume that each K_n is a maximal element of $\mathcal{K}$. We show, by contradiction, that there is $n \in \mathbb{N}$, such that $f(K_n) = G$, this will establish the theorem. So assume $f(K_n) \neq G$ for all $n \in \mathbb{N}$.

Given $n \in \mathbb{N}$, since K_n is maximal in $\mathcal{K}$, K_n does not satisfy (a) of 3.9. Therefore K_n is an arc, $f_{|K_n}$ is an Eulerian path on $f(K_n)$ and $f(K_n)$ is not a maximal element of $\mathcal{G}$. Hence

$$\nu\big(f(K_n)\big) < t_0. \tag{3.10.2}$$

It can be proved that if a subcontinuum H of G admits an Eulerian path and J is an edge of G such that $\operatorname{int}_G(J) \cap H \neq \emptyset$, then $J \subseteq H$. This implies that H is a subgraph of G.

Therefore each $f(K_n)$ is a subgraph of G. Since the number of subgraphs of G is finite, we can assume that the sequence $\{f(K_n)\}_{n=1}^{\infty}$ is constant and that for each $n \in \mathbb{N}$, $f(K_n) = G_0$. Then $\nu(f(K_n)) = t_0$ for each $n \in \mathbb{N}$. This is a contradiction to (3.10.2). This proves the theorem. □

3.11. Theorem. *Let $f : X \to G$ be a Whitney preserving map. Assume that X is arcwise connected, G is nondegenerate and that G is not a simple closed curve. Then f is a homeomorphism.*

Proof. Assume that f is μ, ν-Whitney preserving and that $\nu(G) = 1$. From 3.3, we have that f is a light map. By 2.8, we can assume G is not an arc. Then G has ramification points. To prove the theorem we only need to show that f is one-to-one, we do so by contradiction.

Suppose f is not a one-to-one function.

From 3.10, there exists $K \in C(X)$ such that $f_{|K}$ is almost one-to-one and $f(K) = G$. We consider two cases.

Case 1. $K = X$.

Let $v \in G$ be such that $f^{-1}(v)$ is nondegenerate. Since f is almost one-to-one, $f^{-1}(v)$ is finite. Let $f^{-1}(v) = \{p_1, \dots, p_n\}$ and let $U_1, \dots, U_n$ be pairwise disjoint open sets of X such that, for each $i \in \{1, \dots, n\}$, $p_i \in U_i$. Then $V = G - (f(X - (U_1 \cup \dots \cup U_n)))$ is an open set of G that contains v. Let M be a compact connected neighborhood of v in G such that $M \subseteq V$. Since f is weakly confluent, there exists a subcontinuum A of X such that $f(A) = M$. Note that $A \subseteq U_1 \cup \dots \cup U_n$; hence we can assume that $A \subseteq U_1$. By the Boundary Bumping Theorem (see [6, Theorem 5.4]), there is a nondegenerate subcontinuum B of X such that $p_2 \in B \subseteq U_2 \cap f^{-1}(M)$. Then, for each $b \in B$, $f^{-1}(f(b))$ has at least two points, namely: b and one in A. This contradicts the fact that f is almost one-to-one, so this case is not possible.

Case 2. $K \neq X$.

Since X is arcwise connected, we can construct an arc α_0 satisfying the conditions of 3.7. Then, by 3.7, K is an arc and $f_{|K}$ is an Eulerian path on G.

Let v be a ramification point of G. Since $f_{|K}$ is an Eulerian path on G we have that $f^{-1}(v) \cap K$ has at least two different elements x_1 and x_2. Since f is light, there are two closed disjoint sets E_1 and E_2 of X such that $f^{-1}(v) \subseteq E_1 \cup E_2$, $x_1 \in E_1$, and $x_2 \in E_2$. Let U_1 and U_2 be two open disjoint sets of X such that $E_1 \subseteq U_1$ and $E_2 \subseteq U_2$. Then $V = G - (f(X - (U_1 \cup U_2)))$ is open and $v \in V$.

Let $\varepsilon = \min\{\operatorname{diam}(J)\colon J \text{ is an edge of } G\}$. Let $\delta > 0$ be given by ε and the uniform continuity of f, and such that it satisfies the conditions of 3.1.

Let S be a simple n-od, $n \geqslant 3$, contained in G with the following properties: S is a compact neighborhood of v, $S \subseteq V$ and $\operatorname{diam}(S) < \frac{\varepsilon}{2}$. Let $J_1, \dots, J_n$ be subarcs of G such that $S = \bigcup_{i=1}^{n} J_i$, for all $i \neq j$, $J_i \cap J_j = \{v\}$, and v is an end point of each J_i. For each $i \in \{1, \dots, n\}$, let q_i be the end point of J_i different from v. Then $\operatorname{Fr}_G(S) = \{q_1, \dots, q_n\}$. Since f is weakly confluent, because f is Whitney preserving, there is a subcontinuum R of X such that $f(R) = S$. We can assume that R is a component of $f^{-1}(S)$. Note that $R \subseteq f^{-1}(S) \subseteq U_1 \cup U_2$, so we can assume that $R \subseteq U_1$, then $x_2 \notin R$. Choose a point $x \in R$.

Let $\sigma : [0,1] \to X$ be a one-to-one continuous function such that $\sigma(0) = x$ and $\sigma(1) = x_2$. Let $r_1 = \max\{r \in [0,1]\colon \sigma(r) \in R\}$ and let $r_5 = \min\{r \in [r_1, 1]\colon \sigma(r) \in f^{-1}(v)\}$. Since $\sigma(1) \notin R$, $r_1 < 1$. We claim that $f(\sigma(r_1)) \in \operatorname{Fr}_G(S)$. Otherwise, since $f(\sigma(r_1)) \in f(R) = S$, we would have that $f(\sigma(r_1)) \in \operatorname{int}_G(S)$. Hence there would be a point $r_0 \in (r_1, 1]$ such that $f(\sigma([r_1, r_0])) \subseteq S$. Then $\sigma([r_1, r_0]) \subseteq f^{-1}(S)$ and $\sigma([r_1, r_0]) \cap R \neq \emptyset$. This implies that $\sigma([r_1, r_0]) \subseteq R$, which contradicts the definition of r_1. Therefore $f(\sigma(r_1)) \in \{q_1, \dots, q_n\}$. Assume that $f(\sigma(r_1)) = q_1$. In particular we have that $r_1 < r_5$.

Let $r_2 \in (r_1, r_5)$ be such that $\operatorname{diam}(\sigma([r_1, r_2])) < \delta$, $f(\sigma([r_1, r_2])) \cap \{q_2, \dots, q_n\} = \emptyset$ and $f(\sigma([r_1, r_2]))$ has no ramification points of G. Note that $f(\sigma([r_1, r_2]))$ is an arc with end points q_1 and $f(\sigma(r_2))$ and that it has no ramification points of G. Then, either $f(\sigma([r_1, r_2])) \subseteq J_1$ or $f(\sigma([r_1, r_2])) \cap J_1 = \{q_1\}$. If $\sigma([r_1, r_2]) \subseteq f^{-1}(S)$ were the case, then, since $\sigma(r_1) \in \sigma([r_1, r_2]) \cap R$, $\sigma([r_1, r_2]) \subseteq R$ which is impossible because of the way r_1 was chosen. This argument shows that $f(\sigma([r_1, r_2])) \cap J_1 = \{q_1\}$. Therefore $f(\sigma((r_1, r_2]))$ is a connected subset of G which does not intersect $\operatorname{Fr}_G(S)$ and that it is not contained in S. We have then $f(\sigma((r_1, r_2])) \subseteq G - S$. Since $f(\sigma(r_5)) = v \in S$, we can define $r_4 = \min\{r \in (r_1, r_5]\colon f(\sigma(r)) \in \operatorname{Fr}_G(S)\} = \min\{r \in (r_1, r_5]\colon f(\sigma(r)) \in S\}$. Let $i_0 \in \{1, \dots, n\}$ be such that $f(\sigma(r_4)) = q_{i_0}$. Note that $r_4 < r_5$.

Let $r_3 \in (r_2, r_4)$ be such that $\operatorname{diam}(\sigma([r_3, r_4])) < \frac{\delta}{2}$ and $f(\sigma([r_3, r_4]))$ has no ramification points of G. Then $f(\sigma([r_3, r_4]))$ is an arc with $f(\sigma(r_3))$ and q_{i_0} as its end points and such that it intersects J_{i_0} at q_{i_0}. Hence, the set $L_1 = f(\sigma([r_3, r_4])) \cup J_{i_0}$ is an arc with end points $f(\sigma(r_3))$ and v and it satisfies that $\operatorname{Fr}_G(L_1) = \{f(\sigma(r_3)), v\}$.

We claim that $f(\sigma([r_4, r_5])) \subseteq J_{i_0}$. We show this by contradiction, so assume the statement is false. Since $f(\sigma(r_4)) = q_{i_0} \in J_{i_0}$, it makes sense to define $w_2 = \max\{r \in [r_4, r_5]\colon f(\sigma([r_4, r])) \in J_{i_0}\}$. Let $w_1 \in (r_4, r_5)$ be such that $\operatorname{diam}(\sigma([r_4, w_1])) < \frac{\delta}{2}$ and $f(\sigma([r_4, w_1])) \subseteq L_1 - \{f(\sigma(r_3)), v\}$. By the way δ was chosen and since $\operatorname{diam}(\sigma([r_3, w_1]) < \delta$, we have that $f(\sigma([r_3, w_1]))$ is an arc with end points $f(\sigma(r_3))$ and $f(\sigma(w_1))$. Then $f(\sigma([r_4, w_1])) \subseteq J_{i_0}$. Therefore $w_1 \leqslant w_2$. This shows that $r_4 < w_2$. Now, from the facts that f is Whitney preserving and $\sigma([r_3, w_2])$ is an arc whose image under f is contained in the arc L_1, we have that 2.4 and 2.8 imply that $f_{|\sigma([r_3, w_2])}$ is a one-to-one function. Since $f(\sigma(w_2))$ cannot be an interior point of J_{i_0}, we have that $f(\sigma(w_2)) \in \operatorname{Fr}_G(J_{i_0}) = \{q_{i_0}, v\}$. Given that $q_{i_0} = f(\sigma(r_4))$ and $r_4 < w_2$, we have $f(\sigma(w_2)) = v$. From the definition of r_5 we obtain $w_2 = r_5$. But $f(\sigma([r_4, w_2])) \subseteq J_{i_0}$, hence $f(\sigma([r_4, r_5])) \subseteq J_{i_0}$. Contrary to our assumption. This shows that, indeed, $f(\sigma([r_4, r_5])) \subseteq J_{i_0} \subseteq S$.

By the definition of r_4, $f(\sigma((r_1, r_4))) \cap S = \emptyset$. Hence $f(\sigma([r_1, r_5])) \cap S = \{q_1\} \cup f(\sigma([r_4, r_5])) \subset \{q_1\} \cup J_{i_0}$. Let $L = f(\sigma([r_1, r_5]))$. We can assume that $i_0 \neq 3$. Then $q_3 \notin L$. Since $q_3 \in \operatorname{Fr}_G(S)$, we can take a point $z_0 \in G - (S \cup L)$.

Let $M = R \cup \sigma([r_1, r_5])$. Then M is a subcontinuum of X with the property that $f(M) = S \cup L$. Therefore $f(M) \neq G$. Let $N = R \cup \sigma([r_1, r_4])$. Then N is a subcontinuum of X such that $N \subseteq M$. By the definition of r_1, and since $r_1 < r_4 < r_5$, we have that $\sigma(r_5) \in M - N$. Thus N is a proper subcontinuum of M. Furthermore, $f(N) = S \cup f(\sigma([r_1, r_4])) = (S \cup f(\sigma([r_4, r_5]))) \cup f(\sigma([r_1, r_4])) = f(M)$.

Let $t_1 = \mu(N)$ and $t_2 = \mu(M)$. Then $t_1 < t_2$. Since f is μ, ν-Whitney preserving, there exists $s \in [0,1]$ such that $f(\mu^{-1}(t_2)) = \nu^{-1}(s)$. Then $f(N) = f(M) \in \nu^{-1}(s)$. Hence $f(\mu^{-1}(t_1)) = \nu^{-1}(s)$. Since $z_0 \notin f(M)$, $s < 1$.

We take a point $p_0 \in K$ and a continuous function $\varphi : [0,1] \to C(K)$ such that $\varphi(0) = \{p_0\}$, $\varphi(1) = K$ and such that the following implication holds: if $0 \leqslant a < b \leqslant 1$, then $\varphi(a)$ is a proper subarc of $\varphi(b)$. Since $f(K) = G$, we have that $\nu(f(K)) = 1 > s$. If $\mu(K) \leqslant t_2$, then there exists a subcontinuum K_2 of X such that $K \subseteq K_2$ and $\mu(K_2) = t_2$. This implies that $G = f(K) \subseteq f(K_2) \subseteq G$ and $1 = \nu(G) = \nu(f(K_2)) = s$, which is a contradiction. Hence $t_2 < \mu(K)$. Therefore there are $a_1, a_2 \in [0,1]$ such that $\mu(\varphi(a_1)) = t_1$ and $\mu(\varphi(a_2)) = t_2$. Thus $a_1 < a_2$ and $\varphi(a_1)$ is a proper subarc of $\varphi(a_2)$. Note that

$f(\varphi(a_1)) \subseteq f(\varphi(a_2))$ and $\nu(f(\varphi(a_1))) = s = \nu(f(\varphi(a_2)))$. So $f(\varphi(a_1)) = f(\varphi(a_2))$. Since $f_{|K}$ is almost one-to-one, $f_{|\varphi(a_2)}$ is almost one-to-one. Let T be a subarc of $\varphi(a_2) - \varphi(a_1)$, then, for all $x \in T$, there exists $z \in \varphi(a_1)$ such that $f(x) = f(z)$ (and hence $x \neq z$). This contradicts the fact that $f_{|\varphi(a_2)}$ is almost one-to-one. This shows that Case 2 is impossible.

We have shown that f is one-to-one, therefore it is a homeomorphism. □

As immediate consequences of 3.11 we have the following corollaries.

3.12. Corollary. *Let X be a locally connected continuum and let Y be a nondegenerate finite graph. If $f : X \to Y$ is Whitney preserving function and Y is not a simple closed curve, then f is a homeomorphism.*

3.13. Corollary. *Let K and G be two finite graphs. Assume that G is nondegenerate and that is not a simple closed curve. If there is a Whitney preserving function $f : K \to G$, then K and G are homeomorphic; furthermore f is a homeomorphism.*

3.14. Theorem. *Let $f : X \to G$ be a Whitney preserving function. Assume f is not a homeomorphism, that X contains a dense arcwise connected subset E and that G is nondegenerate. Then G admits an Eulerian path.*

Proof. Since every simple closed curve admits Eulerian paths, we can assume G is not a simple closed curve. From 3.10, there exists $K \in C(X)$ with the property that $K \subseteq E$, $f_{|K}$ is almost one-to-one and $f(K) = G$. By 3.5, K is a finite graph. If $K = X$, by 3.11, f is a homeomorphism, contrary to the hypothesis. Therefore $K \neq X$. Then we can construct an arc α satisfying the conditions on 3.7 to conclude that $f_{|K}$ is an Eulerian path on $f(K) = G$. □

3.15. Theorem. *If G admits an Eulerian path, then there exists a continuum X containing a dense arcwise connected subset E and there exists a Whitney preserving function $f : X \to G$ such that f is not a homeomorphism.*

Proof. We can assume that $G \subseteq [0,1]^3$. Let $\alpha : [0,1] \to G$ be an Eulerian path. For each $n \in \mathbb{N}$, we define $\varphi_n : [n, n+1] \to [0,1]^4$ given by $\varphi_n(u) = (\alpha(u-n), \frac{1}{u})$. Since $\varphi_n(n+1) = (\alpha(1), \frac{1}{n+1}) = (\alpha(0), \frac{1}{n+1}) = \varphi_{n+1}(n+1)$, there exists a continuous function $\varphi : [1,\infty) \to [0,1]^4$ that extends all the functions φ_n. Note that φ is an embedding.

Let $X = (G \times \{0\}) \cup \mathrm{Im}(\varphi)$. Note that X is a compactification of $[1,\infty)$ with $G \times \{0\}$ as remainder. Let $f : X \to G$ be the projection map onto the first three coordinates. That is, $f(x,y,z,w) = (x,y,z)$. We prove that f is Whitney preserving.

Let $\nu : C(G) \to [0,1]$ be any Whitney map such that $\nu(G) = 1$. Let $\mathcal{A} = \mathrm{cl}_{C(X)}(\{A \in C(X){:}\ f(A)$ is a proper subcontinuum of $G\})$. Note that $C(G \times \{0\}) \subseteq \mathcal{A}$. Define $\mu_0 : \mathcal{A} \to [0,1]$ by $\mu_0(A) = \nu(f(A))$. By definition, μ_0 is a continuous function and $\mu_0(\{p\}) = 0$ for all $p \in X$.

Let $A, B \in \mathcal{A}$ be such that $A \subseteq B \neq A$. We show that $\mu_0(A) < \mu_0(B)$. We have two cases.

Case 1. $B \cap (G \times \{0\}) \neq \emptyset$.

In this case, either $B \subseteq G \times \{0\}$ or $G \times \{0\} \subsetneqq B$. If $G \times \{0\} \subsetneqq B$, then $B = (G \times \{0\}) \cup \varphi([M,\infty))$, for some $M \in [1,\infty)$; this, however, contradicts the fact that $B \in \mathcal{A}$. Therefore $B \subseteq G \times \{0\}$. Given that $f_{|G\times\{0\}}$ is one-to-one, $f(A) \subsetneqq f(B)$ and $\mu_0(A) = \nu(f(A)) < \nu(f(B)) = \mu_0(B)$. This ends Case 1.

Case 2. $B \cap (G \times \{0\}) = \emptyset$.

In this case, $A = \varphi([a,b])$ and $B = \varphi([c,d])$ for some $1 \leqslant c \leqslant a \leqslant b \leqslant d < \infty$ and $[a,b] \subsetneqq [c,d]$. Let $e, g \in (c,d)$ be two numbers such that $e < g$, $(e,g) \cap [a,b] = \emptyset$, (e,g) is contained in an interval of the form $[n, n+1]$, for some $n \in \mathbb{N}$, and $\alpha^{-1}(w)$ is degenerate for each $w \in \alpha((e-n, g-n))$. We show that, if $u \in (e,g)$, then $f(\varphi(u)) \in f(B) - f(A)$. Suppose this is not true, thus there exists $v \in [a,b]$ such that $f(\varphi(v)) = f(\varphi(u))$. From the choice of e and g, $v \notin [n, n+1]$, hence we can assume that $v < u$. Then the length of $[v,g]$ is larger than 1 and its image under φ is contained in B. This contradicts the fact that $B \in \mathcal{A}$ and shows that $f(\varphi(u)) \in f(B) - f(A)$. Thus $\mu_0(A) = \nu(f(A)) < \nu(f(B)) = \mu_0(B)$. This ends Case 2.

By Ward's Theorem on extensions of Whitney maps (see [3, Theorem 16.10]), there exists a Whitney function $\mu : C(X) \to [0,1]$ such that $\mu(A) = \frac{1}{2}\mu_0(A)$, for each $A \in \mathcal{A}$.

To complete the proof of the theorem we only need to show that f is μ,ν-Whitney preserving. Let $t \in [0,1]$. We analyze two cases.

Case 1. $t < \frac{1}{2}$.

We will show that $f(\mu^{-1}(t)) = \nu^{-1}(2t)$. First we prove that if $A \in \mu^{-1}(t)$, then $A \in \mathcal{A}$. Let $A \in \mu^{-1}(t)$. If $A \subseteq G \times \{0\}$, then, clearly, $A \in \mathcal{A}$. Now, given that $\mu(G \times \{0\}) = \frac{1}{2}(\mu_0(G \times \{0\})) = \frac{1}{2}$, it does not hold that $G \times \{0\} \subsetneqq A$. Therefore we only need to consider the case when $A = \varphi([a,b])$ for some $1 \leqslant a < b < \infty$. If $f(A)$ is a proper subcontinuum of G, then $A \in \mathcal{A}$. So assume $f(A) = G$. Taking an order arc from $\{\varphi(a)\}$ to A we can find a subcontinuum A_0 of A such that $A_0 \in \mathcal{A}$ and

$f(A_0) = G$. Then $\frac{1}{2} = \frac{1}{2}(\nu(f(A_0))) = \frac{1}{2}(\mu_0(A_0)) = \mu(A_0) \leqslant \mu(A) = t < \frac{1}{2}$, which is impossible. This shows that $f(A) = G$ does not hold. Hence $A \in \mathcal{A}$. Now, by definition, $\nu(f(A)) = 2\mu(A) = 2t$. Hence $f(\mu^{-1}(t)) \subseteq \nu^{-1}(2t)$. To prove the other inclusion, let $B \in \nu^{-1}(2t)$. Since $f_{|G\times\{0\}} : G \times \{0\} \to G$ is a homeomorphism, we can define $A = (f_{|G\times\{0\}})^{-1}(B)$. Clearly, $B = f(A)$ and $A \in \mu^{-1}(t)$. Therefore $f(\mu^{-1}(t)) \supseteq \nu^{-1}(2t)$. This ends Case 1.

Case 2. $\frac{1}{2} \leqslant t$.

Given $A \in \mu^{-1}(t)$, if $f(A)$ is a proper subcontinuum of G, then $A \in \mathcal{A}$; implying $t = \mu(A) = \frac{1}{2}(\nu(f(A))) < \frac{1}{2}$, which contradicts the hypothesis of Case 2. Hence $f(A) = G$. This proves that $f(\mu^{-1}(t)) \subseteq \{G\}$. Therefore $f(\mu^{-1}(t)) = \{G\} = \nu^{-1}(1)$. □

3.16. Remark. Note that by replacing G with S^1 in 3.15, we obtain an example showing that 2.8 cannot be generalized to the case when the image of f is a simple closed curve. Furthermore, 3.15 shows that for any graph admitting an Eulerian path there exists a compactification X of $[1, \infty)$ with remainder G such that the projection $f : X \to G$ is a Whitney preserving map. On the other hand, 2.8, shows that for the unit interval such space and function do not exist. Hence, it is natural to ask the following question.

3.17. Question. For what continua X, such that X is the remainder of a compactification Z of $[1, \infty)$, does there exists a projection map $f : Z \to X$ such that f is a Whitney preserving map?

4. Whitney preserving maps onto S^1

To obtain a characterization of Whitney preserving maps between finite graphs, by 3.13, it only remains to analyze the case when the image is a simple closed curve. In this section we study Whitney preserving maps onto S^1. Throughout this section, $X \approx Y$ means X is homeomorphic to Y.

4.1. Definition. A space Z is said to be *semi-locally-connected at* p (slc at p), if every neighborhood of p contains a neighborhood V of p such that $Z - V$ has only finitely many components. A space Z is said to be semi-locally-connected (slc) if it is semi-locally-connected at every point.

4.2. Proposition. *Let X be an arcwise connected slc continuum and let $f : X \to S^1$ be a Whitney preserving map. If f is hereditarily weakly confluent, then $X \approx S^1$.*

Proof. By Corollary 8.5 of [1], X contains a simple closed curve S.

We show that $X = S$. Assume, by way of contradiction, that $X \neq S$. Then there is $x \in X - S$. Now, since X is arcwise connected, there exists an arc α such that $x \in \alpha$ and $\alpha \cap S = \{p\}$. Let β be a proper subarc of S such that $p \in \beta$ and such that p is neither one of the end points of β. Then $\alpha \cup \beta$ is a simple triod. By 3.3, f is light, then $f(\alpha \cup \beta)$ is nondegenerate. Therefore, there exists a simple triod $T \subset \alpha \cup \beta$ such that $f(T)$ is a proper subarc of S^1. Then, since f is hereditarily weakly confluent and by the remark following 2.4, $f_{|T} : T \to f(T)$ is Whitney preserving. By 2.8, $f_{|T}$ is a homeomorphism. This is a contradiction with T being a simple triod and $f(T)$ being an arc. Therefore $X = S$. □

Since every locally connected continuum is arcwise connected and slc, the previous proposition yields the following corollary.

4.3. Corollary. *Let X be a locally connected continuum and let $f : X \to S^1$ be a Whitney preserving map. If f is hereditarily weakly confluent, then $X \approx S^1$.*

4.4. Remark. The map given in 2.3 is not hereditarily weakly confluent since the restriction of f to $[0, \frac{\pi}{2}]$ is not weakly confluent. This shows the necessity, in 4.2 and in 4.3, for f to be hereditarily weakly confluent. However, if we do not require, in 4.3, f to be hereditarily weakly confluent, we obtain the following theorem, which classifies the locally connected continua X for which there is a Whitney preserving map from X onto S^1.

4.5. Theorem. *Let X be a locally connected continuum and let $f : X \to S^1$ be a Whitney preserving map. Then $X \approx I$ or $X \approx S^1$.*

Proof. Since the only locally connected continua that do not contain simple triods are arcs and simple closed curves, we will prove that X does not contain a simple triod. For this, assume, by way of contradiction, that X contains a simple triod L. Since X is locally connected, by 3.3, $f(L)$ is nondegenerate. Hence there exists a simple triod $T \subset L$ such that $f(T) \neq S^1$. By the fact that $f(T)$ is an arc and by 2.6, $f_{|T} : T \to f(T)$ is Whitney preserving. Therefore, by 2.8, $f_{|T}$ is a homeomorphism; a contradiction due to the fact that T is a simple triod and $f(T)$ an arc.

The previous argument shows that X does not contain a simple triod. Hence, since X is locally connected, $X \approx I$ or $X \approx S^1$. □

The next theorem is a consequence of 3.13 and 4.5, it characterizes all Whitney preserving maps between finite graphs.

4.6. Theorem. *Let K and G be finite graphs and let $f : K \to G$ be a Whitney preserving map. Then either K and G are homeomorphic, or K is homeomorphic to either I or S^1.*

Proof. We divide the proof in two cases.

First assume G is not a simple closed curve. Hence, from 3.13, G is homeomorphic to K.

In the case when G is a simple closed curve, 4.5 implies that G is homeomorphic to either I or S^1. □

The following theorem is a generalization of 4.2 (in the sense that the range of the function is any continuum) furthermore, we prove that the function is a homeomorphism. We do not state 4.2 as a corollary of 4.7 because it is used to prove 4.7.

4.7. Theorem. *Let X be an arcwise connected slc continuum and let $f : X \to Y$ be a Whitney preserving map. If f is hereditarily weakly confluent, then $X \approx Y$. Furthermore, f is a homeomorphism.*

Proof. Since X and Y are continua, to prove that f is a homeomorphism, it is enough to prove that f is one-to-one. Let x and y be two different points of X and let α be an arc from x to y. Then, from 3.3, $f(\alpha)$ is nondegenerate.

Since f is hereditarily weakly confluent, $f_{|\alpha} : \alpha \to f(\alpha)$ is hereditarily weakly confluent implying, from 2.4, that $f_{|\alpha}$ is Whitney preserving.

From 13.70 of [6, p. 310], since f is hereditarily weakly confluent and α an arc, $f(\alpha)$ is either an arc or a simple closed curve. Hence $f_{|\alpha}$ is a map from an arc to an arc or to a simple closed curve, but 4.2 implies that $f(\alpha)$ is an arc. Therefore, by 2.8, $f_{|\alpha} : \alpha \to f(\alpha)$ is a homeomorphism. Therefore $f(x) \neq f(y)$, which implies that f is one-to-one. □

An immediate consequence of the previous theorem is given in the following corollary.

4.8. Corollary. *Every hereditarily weakly confluent Whitney preserving map between locally connected continua is a homeomorphism.*

Again, 2.3 shows the necessity for f to be hereditarily weakly confluent.

4.9. Question. Note that in 4.2 the fact that f is hereditarily weakly confluent implies that X contains a simple closed curve. Hence it is natural to ask, without assuming hereditarily weakly confluence of f, for which arcwise connected slc continua X is there a Whitney preserving map from X onto S^1?

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